Coherent Risk Measures, Valuation Bounds, and (μ, ρ)- Portfolio Optimization*

Stefan Jaschke[a], **Uwe Küchler**[b]

[a]Weierstraß-Institut für Angewandte Analysis und Stochastik, Mohrenstraße 38, 10117 Berlin, Germany, `stefan@jaschke.net`
[b]Humboldt-Universität zu Berlin, Institut für Mathematik, Rudower Chaussee 25 12489 Berlin, Germany, `kuechler@mathematik.hu-berlin.de`

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Abstract. The relation between coherent risk measures, valuation bounds, and certain classes of portfolio optimization problems is established. One of the key results is that coherent risk measures are essentially equivalent to generalized arbitrage bounds, named "good deal bounds" by Cerny and Hodges (1999). The results are economically general in the sense that they work for any cash stream spaces, be it in dynamic trading settings, one-step models, or deterministic cash streams. They are also mathematically general as they work in (possibly infinite-dimensional) linear spaces.

The valuation theory presented seems to fill a gap between arbitrage valuation on the one hand and utility maximization (or equilibrium theory) on the other hand. "Coherent" valuation bounds strike a balance in that the bounds can be sharp enough to be useful in the practice of pricing and still be generic, i.e., somewhat independent of personal preferences, in the way many coherent risk measures are somewhat generic.

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1 Introduction

The mathematical structure of *valuation bounds* in incomplete markets was first established in the context of super-hedging and arbitrage pricing (e.g. (El Karoui and Quenez; 1992; Jouini and Kallal; 1995a,b)). More recently it has been shown that arbitrage bounds can be generalized to "good-deal bounds", and most mathematical results from no-arbitrage pricing just carry over (Cerny

*Most of the motivation came from the landmark paper by Artzner, Delbaen, Eber, and Heath (1998). A substantial amount of the work was done during a stay at Padua University 1998. Special thanks go to Wolfgang Runggaldier for the discussions about the early versions as well as the submitted version of the paper. We like to thank Martin Schweizer and two anonymous referees for valuable comments. We also thank seminar and conference participants from Padua, Berlin, Zürich, Ulm, and Eindhoven for comments.

and Hodges; 1999). Independently, risk measures like "Value at Risk" (VaR) appeared. Deficiencies of (quantile-) VaR led to the question what the characteristics of economically sensible risk measures are. These were developed in the landmark paper on *coherent risk measures* by Artzner et al. (1998). It turns out that the mathematical structure behind coherent risk measures and good-deal bounds is exactly the same. This leads to interesting connections between the two fields, which started from different viewpoints.

This paper works out the relation between risk measures, valuation bounds, and certain classes of portfolio optimization problems. It is economically general in the sense that it provides a common framework for applications in any cash stream spaces, be it in dynamic trading settings, one-step models, or deterministic cash streams. It is mathematically general in the sense that the core results are established for general linear spaces (which may be infinite-dimensional).

The valuation theory presented seems to fill a gap between arbitrage valuation on the one hand and single agent utility maximization or full-fledged equilibrium theory on the other hand. Arbitrage valuation has the advantage that the derived price bounds are completely independent of (estimated) probabilities and personal preferences. Arbitrage bounds can also be robust w.r.t. model misspecification, as in (Brown et al.; 1998). In realistic settings with transaction costs, however, it often leads to weak (wide) price bounds, which are not very useful in practice. Single agent utility maximization can be used to derive much sharper (closer) valuation bounds (as in (Hodges and Neuberger; 1989)). This, however, comes at the cost of the valuation bounds depending on an investor's utility function, his initial position, and his estimate of the probability measure. The valuation bounds that are associated with coherent risk measures strike a balance in that the bounds can be sharp enough to be useful in the practice of pricing and still be generic, i.e. somewhat independent of personal preferences, as well as robust w.r.t. model misspecification and estimation error.

The paper contains some conceptual insights that seem to be new:

Coherent risk measures as functions on a space of random variables (Artzner et al.; 1998) can be generalized to general spaces of economic objects like commodities, delivery contracts, stochastic payment streams, or consumption plans. This is like equilibrium theory, which can be formulated on abstract spaces (Duffie; 1988), but unlike von-Neumann-Morgenstern utilities, which are intricately linked to probability spaces.

Second, there is – except for technical conditions – a one-to-one correspondence between the following economic objects:

(i) "coherent risk measures" ρ,

(ii) cones A of "acceptable risks" or "desirable claims", $(A = \{x \mid \rho(x) \leq 0\})$,

(iii) partial preferences "$x \succeq y$", meaning "x is at least as good as y", $(x \succeq y \iff \rho(x-y) \leq 0)$,

(iv) valuation bounds $\overline{\pi}$ and $\underline{\pi}$ (with $\rho(x) = \overline{\pi}(-x) = -\underline{\pi}(x)$), and

(v) sets K of "admissible" price systems. ($\pi \in K \iff \pi(x) \geq 0$ for all $x \succeq 0$.)

The relation between (iii), (iv), and (v) was established in the arbitrage and super-hedging thread of the literature, whereas the equivalence of (i), (ii), and (v) was established by Artzner et al. (1998).[2] (Section 2.)

Third, the two examples of generating a coherent risk measure "from standard risks" and the "largest coherent risk measure below a given function" from (Artzner et al.; 1998) are shown to be special cases of the same principle: from any set B, define a (possibly coherent) acceptance set A by taking the conic hull $A = \text{cone}(B)$. (Section 3.)

Fourth, a coherent risk measure ρ and a set M of "cash streams available in a market" (i.e., cash streams that can be generated by trading without endowments) define a new risk measure

$$\tilde{\rho}(z) := \inf_{x \in M} \rho(z + x),$$

which is again coherent if M is a cone. If one replaces the natural ordering "$\geq$" among cash streams by a general pre-order "$\succeq$", one gets generalizations of the concepts *arbitrage*, *super-hedging strategy*, and *arbitrage bounds*. The generalized valuation bounds $\overline{\pi}_M$ and $\underline{\pi}_M$ – called *good-deal bounds* by Cerny and Hodges (1999) – correspond to the coherent risk measure $\tilde{\rho}$ by $\tilde{\rho}(x) = \overline{\pi}_M(-x) = -\underline{\pi}_M(x)$. (Section 4.)

Fifth, a series of coherent risk measures and valuation bounds can be defined for *deterministic cash streams*, generalizing an old idea by Hodges and Schaefer (1977). This shows that our generalization of coherent risk measures in fact supports interesting new classes of risk measures. (Section 4.)

Sixth, using the classical (μ, σ)-portfolio optimization theory of Markovitz as a blueprint, (μ, ρ)-portfolio optimization, where ρ is a coherent risk measure, can be considered. The optimization problem defining the "good deal bound" $\overline{\pi}_M$ turns out to be a special case of the (μ, ρ)-optimization problem. It is, however, an extremal point, so that the strategy realizing the hedge in $\overline{\pi}_M$ is usually not done in practice. By introducing a Lagrangian variable, the (μ, ρ)-problem can be transformed into the problem of maximizing $U = \mu - \lambda\rho$, which can be interpreted as a utility function (since μ is linear and ρ is convex). Altogether, each of the following theories is a special case or limit point of the next one: super-hedging and arbitrage theory, coherent risk measures and good-deal bounds, (μ, ρ)-portfolio optimization, and single-agent utility maximization. (Section 6.)

Some minor "mathematical enhancements" to the original theory are made. Where Artzner et al. (1998), Delbaen (1998), and previous works on arbitrage

[2] In their setting, linear price systems can be represented by probability measures and sets of linear price systems are called generalized scenarios.

theory often use topological arguments, we point out that the closedness concepts one really needs are of algebraic nature. The one-to-one correspondence between coherent acceptance sets A and coherent risk measures ρ, for example, hinges on A being *radially closed*, which is a weaker closedness concept and does not correspond to a topology. Also, the separating hyperplane theorem, which underlies the duality between acceptance sets and admissible price systems (and any other strong duality result), can be proven in a purely algebraic setting.

A remark on notation. Important economic concepts that admit different mathematical specializations are put in "quotes", whereas precise mathematical definitions are set in *italic*.

Remarks that the less mathematically inclined reader may skip are set in small, sans-serif paragraphs.

2 Partial Preferences, Acceptable Cash Streams, Valuation Bounds, Risk Measures, and Admissible Price Systems

In this section we are going to show that there is a one-to-one correspondence between certain pre-orders $\succeq$, sets of acceptable cash streams A, valuation bounds $\underline{\pi}, \overline{\pi}$, risk measures ρ, and admissible price systems K.

Let L denote a generic "space of cash streams". It should be possible to form portfolios and be on either side of a contract, so it is natural to assume that L is a *linear space*. Examples of such spaces are:

(i) The space of "stochastic cash streams" on a finite horizon $[0, T]$. Let L_{sm} denote the space of *simple adapted processes* on a probability space $(\Omega, \mathcal{F})$

$$x(t, \omega) = \sum_{i=0}^{n} x_i \, \chi_{E_i}(\omega) \, \chi_{[\tau_i(\omega), T]}(t),$$

with the meaning that x pays the amount x_i at the random time τ_i in the event E_i.[3]

(ii) The space of "deterministic cash streams" on a finite horizon $[0, T]$. Let L_{dm} denote the space of piecewise constant functions

$$x(t) = \sum_{i=1}^{n} x_i \chi_{[\tau_i, T]}(t),$$

with the meaning that x pays the amount x_i at the deterministic time τ_i.

[3] χ_A denotes the indicator function of a set A, and the events $\{\omega | \tau_i(\omega) \leq t\}$ and $E_i \cap \{\omega | \tau_i(\omega) \leq t\}$ are in the information set $\mathcal{F}_t$ of the time t for all $t \in [0, T]$. n is an arbitrary integer.

(iii) The space of "stochastic payments at one period". Let L_{so} denote the space of *simple random variables*

$$x(\omega) = \sum_{i=1}^{n} x_i \chi_{E_i}(\omega)$$

on some probability space $(\Omega, \mathcal{F})$. For finite Ω, this is the case covered in (Artzner et al.; 1998).

The next item we need is a "relatively secure cash stream whose current (time-0) value is 1", which we denote by $\mathbf{1}^L$, or simply $\mathbf{1}$. $\mathbf{1}$ is a "reference cash stream", "benchmark cash stream", or "numeraire cash stream", of which there can be many. A $\mathbf{1}$ in the above spaces can be defined, for example, as

(i) $\mathbf{1}^{L_{sm}}(t, \omega) = \chi_\Omega(\omega)\chi_{[0,T]}(t)$,

(ii) $\mathbf{1}^{L_{dm}}(t) = \chi_{[0,T]}(t)$, and

(iii) $\mathbf{1}^{L_{so}}(\omega) = (1 + r)\chi_\Omega(\omega)$, where r is a riskless interest rate for the interval of the one-step model.

Price systems are real-valued functions $\pi : L \to \mathbb{R}$. If $\pi(x)$ is thought of as a "value" of x before any kind of transaction costs are considered, instead of a market "price", π is naturally a linear function.[4] Let $L^\times$ denote the space of all *linear price systems* on L. $L^\times$ is the *algebraic dual* of L. Call a price system π *(time-0) normalized* if $\pi(\mathbf{1}) = 1$. For example,

(i) an important class of price systems on L_{sm} is generated by pairs of numeraire processes $N_t > 0$ (e.g. the value of a money market account) and probability measures Q by

$$\pi_{N,Q}(x) = \mathrm{E}_Q \left\{ \int_0^T \frac{N_0}{N_t} dx_t \right\};$$

(ii) price systems on L_{dm} are nothing else than *term structures of interest rates*, which can be represented by discount factors $v(t)$ for every time to maturity t, such that

$$\pi_v(x) = \int_0^T v(t) dx_t;$$

[4] This does not mean that transaction costs are not considered. It only means that our *linear price systems* π are always defined on the whole space L and $\pi(x)$ is to be seen as a "value" rather than a "market price". "Market prices" are usually not linear and may not be defined on the whole space L.

(iii) price systems on L_{so} are equivalent to finitely additive set functions Q on $(\Omega, \mathcal{F})$ by

$$\pi_Q(x) = \mathrm{E}_Q\{x\} = \int_\Omega x(\omega) Q(d\omega).$$

Note that the business of pricing OTC-contracts consists in estimating a normalized price system $\hat{\pi}$. $\hat{\pi}(x)$ is then the (expected) present value of the cash stream x. So, the concept of a time-0 normalized linear price system is essentially the same as the "present value principle".

A relation $\succeq$ on L is a *pre-order* if

$(a)\quad x \succeq x$ and

$(b)\quad x \succeq y, y \succeq z \implies x \succeq z$

for all $x, y, z \in L$. A pre-order is a *vector ordering* if

$(c)\quad x \succeq y \iff x - y \succeq 0$ and

$(d)\quad x \succeq 0,\, \alpha \geq 0 \implies \alpha x \succeq 0$

for all $x, y \in L$. It is well-known that there is a one-to-one correspondence between vector orderings $\succeq$ and cones[5] A by $x \succeq y \iff x - y \in A$.

A *natural vector ordering* $\geq$ on cash stream spaces is given by "$x \geq 0$ if every possible single payment of x is nonnegative". The cone of nonnegative cash streams is denoted by L^+. In the above spaces this means that

1. L^+_{sm} (L^+_{dm}) is the set of nondecreasing processes (functions) x with $x_0 \geq 0$, and

3. L^+_{so} is the set of nonnegative random variables.

If now $z \succeq 0$ has the meaning that "z is in some sense at least as good as the zero cash stream" or "z is an acceptable risk", or "z is a desirable claim",[6] then

$$\overline{\pi}(z) = \inf\{\alpha \,|\, \alpha\mathbf{1} \succeq z\}$$

can be considered an "upper bound" for the price (or the insurance premium) of the cash stream z. Analogously,

$$\underline{\pi}(z) = \sup\{\alpha \,|\, \alpha\mathbf{1} \preceq z\}$$

can be considered a "lower bound" for the price of z. The function ρ defined by

$$\rho(z) = \inf\{\alpha \,|\, \alpha\mathbf{1} + z \succeq 0\}$$

[5] A is a *cone* if $A + A \subseteq A$ and $\alpha A \subseteq A$ for all $\alpha \geq 0$.

[6] The indicated economic meaning of $\mathbf{1}$, L^+ and $\succeq$ suggests the inclusion $\mathbf{1} \in L^+ \subseteq \{x | x \succeq 0\}$. We will still explicitly say when we assume that, though.

can be considered a "risk measure". Obviously, $\rho(z) = \overline{\pi}(-z) = -\underline{\pi}(z)$ for all $z \in L$.[7]

A set A is said to be *absorbing* if for every $x \in L$ exists an $\alpha > 0$ such that $\alpha^{-1}x \in A$. The *radial interior* of A is the set of all $x \in A$ such that $A - x$ is absorbing. A set A is *radially open*, if it equals its radial interior. The complements of radially open sets are called *radially closed*.

Call

$$\rho_A(z) := \inf\{\alpha \,|\, \alpha\mathbf{1} + z \in A\} \qquad (\inf\emptyset := \infty) \tag{1}$$

the *risk measure associated with* a set A and

$$A_\rho := \{z \,|\, \rho(z) \leq 0\}$$

the *acceptance set associated with* a function ρ. It is easily verified that $\rho_{A_\rho} = \rho$ holds for all functions into the extended real line[8] $\rho : L \to \overline{\mathbb{R}}$ with the *translation property*

$$(T_\rho) \quad \rho(\alpha\mathbf{1} + z) = \rho(z) - \alpha \quad \forall\alpha \in \mathbb{R}, \forall z \in L.$$

Obviously, $A \subseteq A_{\rho_A}$. The translation property

$$(T_A) \quad x \in A, \alpha \geq 0 \implies x + \alpha\mathbf{1} \in A.$$

ensures that the set $\{\alpha \,|\, \alpha\mathbf{1} + x \in A\}$ is either empty, the whole line, or an interval of the form $[\alpha, \infty)$ or (α, ∞). If A is also radially closed, the last form of the interval cannot occur, the infimum in (1) is attained or $\pm\infty$[9], and $A = A_{\rho_A}$ holds. Clearly, cones A have the translation property if and only if $\mathbf{1} \in A$. If A is a cone, $\rho_A(0) = 0$ if and only if $-\mathbf{1} \notin A$ and $\mathbf{1} \in A$. If A is a cone containing $\mathbf{1}$, then $\rho_A < \infty$ if and only if $\mathbf{1}$ is in the radial interior of A, and $\rho_A > -\infty$ if and only if $-\mathbf{1}$ is in the radial interior of the complement of A. If A is a cone, the associated risk measure ρ_A is convex[10] and positively homogeneous ($\rho(\alpha x) = \alpha\rho(x)$ for all $x \in L, \alpha > 0$). Conversely, if ρ has these two properties, A_ρ is a cone. In summary, there is a one-to-one correspondence between radially closed cones A containing $\mathbf{1}$, and positively homogeneous, convex functions ρ that have the translation property.

[7] Although mathematically trivial, it seemed – at the time of writing – not widely recognized that any valuation principle that yields price bounds also induces a risk measure and vice versa.

[8] $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$

[9] with the convention $\inf\emptyset = +\infty$, $\inf\mathbb{R} = -\infty$

[10] In the definition of convexity *inf-addition* $\infty - \infty = \infty$ is used (as in Rockafellar and Wets (1998)).

The relation between an acceptance set and its associated risk measure is similar to the relation between a convex, absorbing set U and its Minkowski functional p_U. In fact, the risk measure ρ_A associated with a cone A can be represented in terms of the Minkowski functional.

Proposition 1 *If A is a cone, $\mathbf{1}$ is in the radial interior of A and $-\mathbf{1}$ is in the radial interior of the complement of A, then ρ_A equals $-p_{L\setminus(A+\mathbf{1})}$ on the radial interior of A, $p_{A-\mathbf{1}}$ on the radial interior of the complement of A, and 0 on the radial boundary of A.*

If $\succeq$ is a vector ordering, an *order interval* $[x, y]$ is defined as the set $\{z \in L \,|\, x \preceq z \preceq y\}$. An element $\mathbf{1} \in L$ is called an *order unit* if $[-\mathbf{1}, \mathbf{1}]$ is absorbing (Schaefer; 1966). It is easily verified that $\mathbf{1}$ is an order unit if and only if $\mathbf{1}$ is in the radial interior of $A = \{x | x \succeq 0\}$.

Now we collect the conditions for the four "economically equivalent" objects $\succeq$, A, ρ, and $(\overline{\pi}, \underline{\pi})$ to be *mathematically equivalent* in formal definitions. We call a vector ordering $\succeq$ on L a set of *coherent partial preferences* if

(Cl) $\{x \,|\, x \succeq 0\}$ is radially closed and

(M) $x \geq 0 \implies x \succeq 0$

for all $x, y \in L$. We say that a set $A \subseteq L$ is a *coherent acceptance set* if

(C, PH) A is a cone,

(Cl) A is radially closed,

(T) $\mathbf{1} \in A$, and

(M) $L^+ \subseteq A$.

Usually, L^+ will be radially closed and contain $\mathbf{1}$, so that it is itself a coherent set of acceptable cash streams. Call a function $\rho : L \to \overline{\mathbb{R}}$ a *coherent risk measure* if

(C) ρ is convex,

(PH) ρ is positively homogeneous (linear in scale): $\rho(\alpha x) = \alpha\rho(x)$ for all $\alpha > 0$, $x \in L$,

(T) $\rho(x + \alpha\mathbf{1}) = \rho(x) - \alpha$ for all $\alpha \in \mathbb{R}, x \in L$, and

(M) $x \in L^+ \implies \rho(x) \leq 0$.

Finally, $(\underline{\pi}, \overline{\pi})$ is a pair of *coherent valuation bounds* if $\overline{\pi}(-z) = -\underline{\pi}(z)$ for all $z \in L$ and $-\underline{\pi}$ is a coherent risk measure.

Corollary 2 *There is a one-to-one correspondence between coherent partial preferences $\succeq$, coherent sets of acceptable cash streams A, coherent risk measures ρ, and pairs $(\underline{\pi}, \overline{\pi})$ of coherent valuation bounds. The correspondence remains valid if the monotonicity condition (M) is removed from each of the definitions.*

The monotonicity condition (M) ensures that $\underline{\pi}(x) \geq 0$ for all $x \geq 0$. We call $(\underline{\pi}, \overline{\pi})$ *weakly relevant* if also $\overline{\pi}(x) \leq 0$ for all $x \leq 0$. The equivalent condition on the acceptance set is $A \cap (-L^+) = \{0\}$. (This is the relevance condition in (Artzner et al.; 1998).) We call coherent valuation bounds $(\underline{\pi}, \overline{\pi})$ *strongly relevant* if $\underline{\pi}(x) \geq 0$ and $\overline{\pi}(x) \leq 0$ imply $x = 0$. The equivalent conditions on the other representations are

$$(R_{\succeq}) \quad x \succeq y \wedge y \succeq x \implies x = y, \text{ i.e. } \succeq \text{ is } \textit{antisymmetric}$$
$$(R_A) \quad A \cap (-A) = \{0\}, \text{ and}$$
$$(R_\rho) \quad \rho(x) \leq 0 \wedge \rho(-x) \leq 0 \implies x = 0.$$

It can be argued (as by Delbaen (1998)) that it doesn't make much sense economically to allow $\rho(x) = -\infty$. A consequence of our earlier observations is that a coherent acceptance set A defines a risk measure $\rho_A > -\infty$ if and only if $-\mathbf{1} \notin A$. This is automatically fulfilled by weakly relevant acceptance sets if $\mathbf{1} \in L^+$.

A semi-norm that is naturally associated with a coherent risk measure ρ is given by

$$p(x) := \rho(-x) + \rho(x) = \overline{\pi}(x) - \underline{\pi}(x).$$

(The sum of two convex functions is convex and p is obviously absolutely homogeneous.)
An ordered linear space $(L, \succeq)$ is a *vector lattice* if for any pair $(x, y) \in L \times L$ the infimum and the supremum of x and y (w.r.t. $\succeq$) exist. Then one can define

$$|x| = \sup(-x, x),$$

and observe that

$$|\lambda x| = |\lambda|\,|x|$$
$$|x + y| \preceq |x| + |y|.$$

(Schaefer; 1966, p.207) From these properties it follows that for any coherent risk measure ρ on a vector lattice

$$p(x) := \rho(-|x|)$$

is another semi-norm.

A large part of the theory of (mathematical) finance is based on the ability to describe sets of cash streams A through "dual sets" $K \subseteq L^\times$ of price systems in the sense that

$$x \in A \iff \pi(x) \geq 0 \text{ for all } \pi \in K. \tag{2}$$

For infinite-dimensional spaces L, the algebraic dual $L^\times$ is not a handy space. It is often more convenient to consider only price systems from a *total subspace* of $L^\times$.[11] L' is a *total subspace* of $L^\times$ if $\pi(x) = 0$ for all $\pi \in L'$ implies $x = 0$. The $\sigma(L')$-topology on L is the coarsest topology for which the linear functionals $x \mapsto \pi(x)$ are continuous for all $\pi \in L'$. Similarly, the $\sigma(L)$-topology on L' is

[11] For example, one wants to use σ-additive measures instead of finitely additive set functions to be able to use probability theory in the analysis of L_{so} and L_{sm}.

the coarsest topology for which the linear functionals $\pi \mapsto \pi(x)$ are continuous for all $x \in L$.

Let

$$A^* := \{\pi \in L' \mid \pi(x) \geq 0 \quad \forall x \in A\}$$

denote the *right polar cone in* L' of a cone $A \subseteq L$ and

$$^*K := \{x \in L \mid \pi(x) \geq 0 \quad \forall \pi \in K\}$$

the *left polar cone* of a cone $K \subseteq L'$. If A is an acceptance set, we call A^* the associated *set of admissible price systems.*

It is easily seen that $A \subseteq {}^*(A^*)$ generally holds. The condition that (2) holds with $K = A^*$ can be written as $A = {}^*(A^*)$. It turns out that this is the case if and only if the cone A is $\sigma(L')$-closed. This result and its consequences for the optimization problem that is dual to (1) is formulated in the following theorem.

Theorem 3 (Duality Theorem) *Let A be a cone that contains $\mathbf{1}$, $K = A^*$ its right polar cone, and $\rho = \rho_A$ its associated risk measure. Then*

(i) $A \subseteq {}^*K$. The equality holds if and only if A is $\sigma(L')$-closed.

(ii) If the set of normalized admissible price systems

$$D := \{\pi \in K \mid \pi(\mathbf{1}) = 1\}$$

is not empty then -$\mathbf{1}$ is not in A. If A is $\sigma(L')$-closed, the converse also holds.

(iii)

$$\rho(x) \geq \begin{cases} +\infty & \text{if } \exists \pi \in K : \pi(x) < 0 \wedge \pi(\mathbf{1}) = 0 \\ \sup\limits_{\pi \in D} \pi(-x) & \text{else} \end{cases} \tag{3}$$

or, more compactly written

$$\rho(x) \geq \sup_{\pi \in K} \pi(-x)/\pi(\mathbf{1}) \tag{4}$$

with the convention $0/0 = -\infty$. If $\mathbf{1}$ is in the radial interior of A, this can be simplified to

$$\rho(x) \geq \sup_{\pi \in D} \pi(-x) \tag{5}$$

with the convention $\sup \emptyset = -\infty$. If A is $\sigma(L')$-closed, equality holds in (3), (4) and (5).

Proof. (i) This is the bipolar theorem. (See the last section.)
(ii) The first statement is trivial. The other implication is a consequence of $A = {}^*K$.
(iii) $A \subseteq {}^*K$ implies

$$\rho(x) \geq \inf\{\alpha \mid \pi(\alpha \mathbf{1} + x) \geq 0 \ \forall \pi \in K\} \tag{6}$$

and $A = {}^*K$ implies the equality in (6). (3), (4) and (5) are simple consequences of (6). □

In light of this theorem we call a coherent risk measure (and its equivalent representations) $\sigma(L')$-*closed* if the associated acceptance set is $\sigma(L')$-closed. Call $K \subseteq L'$ a *coherent set of admissible price systems* if

- (C, PH) K is a cone,
- (Cl) K is $\sigma(L)$-closed, and
- (M) $x \in L^+ \implies \pi(x) \geq 0$ for all $\pi \in K$.

Corollary 4 *There is a one-to-one correspondence between $\sigma(L')$-closed coherent (i) partial preferences $\succeq$, (ii) acceptance sets A, (iii) risk measures ρ, (iv) pairs $(\underline{\pi}, \overline{\pi})$ of coherent valuation bounds, and (v) coherent sets of admissible price systems K. The correspondence remains valid if the monotonicity condition (M) is removed from each of the definitions.*

The following proposition provides sufficient conditions for an acceptance set to be closed in some $\sigma(L')$-topology:

Proposition 5 *Given a radially closed cone A containing 0.*

(i) If A is radial at some point, A is $\sigma(L^\times)$-closed and $A = {}^*(A^*)$ in $L^\times$,

(ii) If A has an interior point in a vector topology $\mathcal{T}$, the linear forms in A^* are $\mathcal{T}$-continuous.

(Grothendieck; 1992, p.52). (See the last section.)

Many works on no-arbitrage pricing (including (Cerny and Hodges; 1999)) start with a *topological* vector space L and require price systems to be from the *strong dual* L^*. This makes the notion of price systems as well as duality theorems appear to depend on topological properties, where in fact they do not. In this section, we proposed an alternative route. Consider all functionals from the algebraic dual $L^\times$. The duality $A = {}^*(A^*)$ holds for all coherent acceptance sets A which are radial at some point. In fact, the **1**-cash streams in the spaces L_{sm}, L_{dm}, and L_{so} as defined above are in the radial interior of the respective positive cones, so that any coherent acceptance set A on these spaces is $\sigma(L^\times)$-closed and a strong duality result holds. The admissible price systems A^* – these are the only ones actually used – are automatically continuous in every topology in which A has an interior point. Alternatively, one can consider a space of price systems L' suggested by *economic* considerations (as long as it defines a total subspace of $L^\times$) and then use the topology $\sigma(L')$ for duality results.

A completely different story is if one wants to define coherent risk measures on the space of all random variables $L^0(\Omega)$, if it is infinite-dimensional. The $\sigma-$additive measures on Ω, which one would like to take as representations of price systems do not define linear functionals that are defined on the whole space L^0. In other words, there is no easily representable total subspace of the algebraic dual of L^0. One work-around is to first define the risk measure on L^∞ and then extend it through a limiting procedure to L^0 (Delbaen; 1998). Another elegant approach is to allow price systems to take the value ∞ and use a more general bipolar theorem then the one we used here (Brannath and Schachermayer; 1998).

3 The Construction of Risk Measures from Standard Risks and Scenarios

If $\mathcal{Y} \subseteq L$ is a set of "standard risks" and, say a bank regulating authority, assigns risk numbers $\psi(y)$ to these cash streams – from section 2 we know that this is equivalent to determining "lower bounds" $-\psi(y)$ for the value of the cash streams y – then $A_0^\psi = \{y + \psi(y)\mathbf{1}\}_{y\in\mathcal{Y}}$ is a set of acceptable cash streams. Positive multiples and convex combinations of acceptable risks should also be acceptable, hence the *finitely generated cone*

$$\mathrm{cone}(A_0^\psi) := \{\sum_{i=1}^n \lambda_i a_i \,|\, \lambda_i \geq 0, a_i \in A_0^\psi, n \in \mathbb{N}\}$$

defines an "acceptance set". Then the risk measure associated with the set

$$A^\psi = \mathrm{cone}(A_0^\psi) + L^+$$

is

$$\rho^\psi(z) = \inf_{\alpha,h}\{\alpha \,|\, z + \alpha\mathbf{1} - \sum_y h_y(y + \psi(y)\mathbf{1}) \geq 0, h_y \geq 0\},$$

where the infimum is taken over all families $\{h_y\}_{y\in\mathcal{Y}}$ with only finitely many elements being non-zero.

Proposition 6 *(compare (Artzner et al.; 1998, Prop.4.2,p.18))*
For any function $\psi : \mathcal{Y} \to \mathbb{R}$, ρ^ψ is the largest coherent risk measure such that $\rho^\psi \leq \psi$ on $\mathcal{Y}$.

Proof. For every two sets A_1 and A_2 with the translation property (T_A) $A_1 \subseteq A_2$ holds if and only if $\rho_{A_1} \geq \rho_{A_2}$. The proposition then follows from the fact that A^ψ is the smallest cone containing A_0^ψ and L^+. It is easily checked that $\mathrm{cone}(A \cup L^+) = \mathrm{cone}(A) + L^+$ for any set A. It is well known that cone(A) is the smallest cone containing A. (One has to show (i) that cone(A) is a cone and (ii) that cone(A) is contained in any cone containing A. The first property is obvious and the second is proved by induction. (Grothendieck; 1992, p.47).) □

Example. Consider the space of deterministic cash streams on a fixed grid of times $0 = \tau_0 < \tau_1 < \ldots < \tau_n$. Represent cash streams by vectors z of single payments z_j and price systems by vectors v of discount factors v_j for the times τ_j. Assume upper and lower bounds $(\underline{f}_j, \overline{f}_j)$ for the forward rates of the intervals $[\tau_j, \tau_{j+1}]$. This defines "standard acceptable risks" of the form

$$\begin{aligned} H_{(j,.)} &= (0, \ldots, 0, -1, 1 + \overline{f}_j, 0, \ldots, 0), \\ H_{(n+j,.)} &= (0, \ldots, 0, 1, -(1 + \underline{f}_j), 0, \ldots, 0). \end{aligned}$$

The corresponding acceptance set is

$$\begin{aligned} A &= \{H'h | h \geq 0\} + L^+, \\ &= \{z \,|\, z - H'h \geq 0, h \geq 0\}, \end{aligned}$$

its polar cone

$$\begin{aligned} K &= \{v \,|\, Hv \geq 0, v \geq 0\}, \\ &= \{v \,|\, \underline{f}_j \leq v_j / v_{j+1} - 1 \leq \overline{f}_j, v \geq 0\}, \end{aligned}$$

and the corresponding coherent risk measure is given by the linear programs

$$\rho(z) = \min_{p,h} \{p \,|\, (p, 0, \ldots, 0) - H'h + z \geq 0, h \geq 0, p \text{ free}\}, \tag{7}$$

$$= \max_{v} \{-z'v \,|\, Hv \geq 0, v \geq 0, v_0 = 1\}. \tag{8}$$

The corresponding valuation bounds were used by Hodges and Schaefer (1977) in their analysis of the British gilts market. This the earliest application of coherent valuation bounds we are aware of. (The formulation of the same idea in the spaces L_{dm} and L_{sm} is obvious.)

An alternative way of constructing a risk measure is to finitely generate the set of admissible price systems by $K = \mathrm{cone}(\{\pi_i\}_{i \in I})$, where we assume that the price systems are normalized $(\pi_i(\mathbf{1}) = 1)$ and non-negative $(x \in L^+ \implies \pi_i(x) \geq 0)$.

Proposition 7 *(Compare (Artzner et al.; 1998, Prop 3.1, p.10).) If a possibly infinite family $\{\pi_i\}_{i \in I}$ of normalized, non-negative price systems is given,*

$$\rho(x) := -\inf_{i \in I} \pi_i(x) \tag{9}$$

defines a coherent risk measure.

Proof. π_i non-negative implies the monotonicity of ρ, π_i normalized implies the translation property, and π_i linear implies that ρ is positively homogeneous and convex. □

A special case of this is when a set of scenarios $\mathcal{T} \subseteq \Omega$ is given, which defines a set of normalized price systems on the space L_{sm} by

$$\pi_\omega(x) = \int_0^T \frac{N_0(\omega)}{N_t(\omega)} dx_t(\omega),$$

which is the ex-post discounted value of x under the scenario $\omega \in \mathcal{T}$. (N_t is the price process of a numeraire.) This was developed into a powerful methodology and implemented by Studer (1997) under the name of *Maximum Loss Optimization*.

It is remarkable that all the general methods of constructing coherent risk measures boil down to taking the conic hull of a set.

4 Marketed Cash Streams, Arbitrages, and Good Deals

In this section we will introduce "good deals", which are a natural generalization of "arbitrages". The key result is that the good-deal bounds – the valuation bounds derived from market prices and the assumption that no good deals exist – are again coherent valuation bounds.

In the following, we denote by M the set of "cash streams that can be generated by trading with zero initial endowment", which we call *available in the market* and assume that it is a cone. (M plays here the same role as the final pay-outs of *self-financing strategies* with initial endowment of 0 do in the classical theory (Harrison and Pliska; 1981).)

Example. Consider a set of deterministic cash streams $\{Y_i\}_{i=1,\dots,m}$ that are traded at bid and asked prices $\underline{P}_i$ and $\overline{P}_i$, respectively. Then the buying of the cash stream Y_i at the asked price generates the net cash stream $C_i := -\overline{P}_i\mathbf{1}+Y_i$, whereas selling at the bid price generates the net cash stream $C_{m+i} := \underline{P}_i\mathbf{1}-Y_i$. Such net cash streams we call "available in the market". If $\{C_1,\dots,C_{2m}\}$ is a set of cash streams available in the market, then any cash stream of the form $\sum_{i=1}^{2m} x_iC_i$ can be generated by buy-and-hold positions $x_i \geq 0$ in the net cash streams C_i. If there are no trading constraints, the set of cash streams available in the market is

$$M = \text{cone}(\{C_1,\dots,C_{2m}\}).$$

If the portfolios x are restricted to a cone other than $(R^{2m})^+$, then the set of cash streams that can be generated by such portfolios still is a cone.

Example. If $\underline{P}_i(t), \overline{P}_i(t)$ are the bid and asked prices at time t of a security that provides the (dividend) cash stream $D_i \in L_{sm}$, dynamic trading allows to generate cash streams of the form

$$C^{i,t,F}(s,\omega) = (D_i(s) - D_i(t) - \overline{P}_i(t))\chi_{[t,T]}(s)\chi_F(\omega) \tag{10}$$

by "buying security i at time t in the event F and holding it till time T", where the event F is in the information set of time $t-1$. (We assumed here that the "final value" of the security i is paid out as $D_i(T) - D_i(T-)$.) The analog cash streams formed by selling are

$$C^{i,t,F}(s,\omega) = -(D_i(s) - D_i(t) - \underline{P}_i(t))\chi_{[t,T]}(s)\chi_F(\omega). \tag{11}$$

If the number of traded securities, the number of trading times, and the probability space Ω are finite, then the set of cash streams available in the market is again a finitely generated cone, generated by all cash streams of the form (10) and (11).

Fix a coherent acceptance set A. A cash stream $x \in M$ represents an opportunity to "get something good for free, where the good part may or may not come some time in the future" if $x \in A$ and $x \neq 0$. Such a cash stream we will call a *good deal of the first kind*[12]. $x \in M$ represents an opportunity to get a "cash-and-carry good deal" if there exists an $\alpha > 0$ such that $x - \alpha\mathbf{1} \in A$. Such a cash stream we will call a *good deal of the second kind.* Since A and M are cones, the absence of good deals of the second kind is equivalent to $\mathbf{1} \notin M - A$. If $A = L^+$, good deals are called *arbitrages* (Ingersoll; 1987). If $-\mathbf{1} \notin A$ then any good deal of the second kind is also a good deal of the first kind.

We believe that the second concept is much more important practically. If a good deal of the first kind is not a good deal of the second kind, then it is not possible to materialize the "maybe something in the future". Since arbitrage transactions in practice always involve some risks or costs that can not be mirrored in a model, arbitrageurs will act only if the "maybe something in the future" is substantial enough, meaning that it can be somehow expressed in units of $\mathbf{1}$. In the following, we only talk about good deals, meaning good deals of the second kind.

The marketed cash streams M and the coherent acceptance set A induce the *good-deal bounds*

$$\overline{\pi}_M(z) \quad = \quad \inf_{\alpha\in\mathbb{R}, x\in M} \{\alpha \,|\, x + \alpha\mathbf{1} - z \in A\}, \text{ and} \tag{12}$$

$$\underline{\pi}_M(z) \quad = \quad \sup_{\alpha\in\mathbb{R}, x\in M} \{\alpha \,|\, x - \alpha\mathbf{1} + z \in A\}. \tag{13}$$

[12] It seems that Cochrane and Saá-Requejo (1996) were the first to use the term "good deals" with the meaning of "generalized arbitrages", and "good-deal bounds" with the meaning of "range of prices consistent with the absence of good deals". Their good deals defined in terms of the Sharpe ratio (on sets of random variables) usually fail on the monotonicity condition and are thus not good deals in our sense. (On infinite-dimensional probability spaces Ω, the cone $\{\mu(X)/\sigma(X) \geq h\}$ contains L^+_{so} only if $h \leq 0$.) Cerny and Hodges (1999), however, used the term "good deals" essentially in the same sense as our "good deals of the first kind", which encourages us to also use that term.

$[\underline{\pi}_M(z), \overline{\pi}_M(z)]$ is the interval of those prices for the cash stream z that are consistent with the absence of good deals. If, for example, someone else is willing to buy the cash stream z for a price $P > \overline{\pi}_M(z)$, there exist a "hedge" $x^* \in M$ and a "price" $\alpha^* < P$ with $x^* + \alpha^* \mathbf{1} - z \in A$. Hence we can sell z at the price P, run the trading strategy that generates x^* and the resulting cash stream $x^* + P\mathbf{1} - z$ is a good deal. Analogously, a good deal can be formed if a cash stream z is offered for a price less then $\underline{\pi}_M(z)$.

Proposition 8 *If A is a coherent acceptance set and M is a cone, then $(\overline{\pi}_M, \underline{\pi}_M)$ defined by (12), (13) is a pair of coherent valuation bounds.*

Proof. It is easily seen that $\overline{\pi}_M(x) = -\underline{\pi}_M(-x)$ and $\rho = -\underline{\pi}_M$ is the risk measure associated with the acceptance set $A - M$. $A - M$ is a cone and contains L^+, so $\rho_{A-M} = -\underline{\pi}_M$ is a coherent risk measure. (If $A - M$ is radially closed then the relation between $A - M$ and ρ_{A-M} is one-to-one.) □

Moreover,

$$\rho_{A-M}(z) = \inf_{x \in M} \rho_A(x + z).$$

We call

$$K_M := A^* \cap (-M)^*$$

the *set of consistent*[13] *price systems.*

$$D_M := \{\pi \in K_M \,|\, \pi(\mathbf{1}) = 1\}$$

is the *set of normalized consistent price systems.*

Although the mathematical structure of the good deal bounds $(\overline{\pi}_M, \underline{\pi}_M)$ is the same as that of the "raw" valuation bounds $(\overline{\pi}, \underline{\pi})$, their economic meaning is very different. The bounds $(\overline{\pi}, \underline{\pi})$ are usually defined to yield bounds that are somewhat independent of current market prices, whereas $(\overline{\pi}_M, \underline{\pi}_M)$ bounds prices of traded securities at least to their bid and asked prices. In terms of acceptance sets, A usually is "a bit more than L^+", whereas $A - M$ usually is "a bit less than a half-space". In terms of normalized price systems, D is the set of "all reasonable price systems", whereas D_M is a "small neighborhood" around a "fitted" price system.

Example. (continued) Consider again the set of deterministic cash streams on a grid $0 = \tau_0 < \ldots < \tau_n$. Let H denote the matrix whose rows are the "standard acceptable risks" defined in the Hodges-Schaefer-example. Let C denote the matrix whose rows are the "available cash streams" from the previous example. The upper good-deal bound is then given by the linear programs

$$\overline{\pi}_M(z) = \min_{p,h,x} \{p \,|\, (p, 0, \ldots, 0) + C'x - H'h \geq z, h \geq 0, x \geq 0, p \text{ free}\}, \quad (14)$$

$$= \max_{v} \{z'v \,|\, Cv \leq 0, Hv \geq 0, v \geq 0, v_0 = 1\}. \quad (15)$$

[13] "linear price systems consistent with the market prices implicit in M"

It is apparent from the primal problem (14) that for any "standard acceptable cash stream" H_j the cash stream $-H_j$ can be interpreted as "auxiliary cash stream" which is "available in the market", even if not actually traded. The variable x_i represents a portfolio position in the cash stream C_i available in the market, while h_j can be interpreted as a portfolio position in the auxiliary cash stream $-H_j$. So we have in fact three interpretations of the matrix H:

(i) $\overline{f}_i, \underline{f}_i$ are assumed bounds of the future interest in the interval $[\tau_i, \tau_{i+1}]$ and the auxiliary variables h correspond to cash management in the future. (This was the interpretation in (Hodges and Schaefer; 1977).)

(ii) $\overline{f}_i, \underline{f}_i$ are bounds for the forward rates of the interval $[\tau_i, \tau_{i+1}]$ and h is a transaction in FRAs at time 0.

(iii) The dual constraints

$$v_{\tau_{i+1}} \underline{f}_i \leq v_{\tau_i} - v_{\tau_{i+1}} \leq v_{\tau_{i+1}} \overline{f}_i$$

bound the first "derivative" of the (discrete) discount function.

All three interpretations can be used to exogenously define "standard acceptable risks". The last interpretation lends itself to be generalized to *higher order conditions* on the term structure (Jaschke; 1999).

To see in which sense D_M is a "neighborhood" of a fitted term structure assume that the bid and asked prices are $P_i \pm \epsilon$. Then the condition "consistent with observed market prices", $Cv \leq 0$, becomes $|P_i - y_i' v| \leq \epsilon$, for all i. Many non-parametric term structure estimators can be written as the optimal solution of

$$\min_{v \in D} \{ ||P - Yv|| + \lambda \, \text{roughness}(v) \}$$

for some roughness measure and some norm. (If $\text{roughness}(v) = \int ([\log v_t]'')^2 dt$, the estimator is a cubic *smoothing spline*.) If the pricing error $P - Yv$ is minimized in the $\infty-$norm, the estimator is the smoothest term structure in the set of consistent price systems

$$D_M = \{ v \,|\, ||P - Yv||_\infty \leq \epsilon, v \in D \},$$

for some ϵ related to λ. (The connection between arbitrage theory under proportional transaction costs and minimax-fitting also holds for dynamic trading, which is implicit in (Jouini and Kallal; 1995b) and made more explicit in (Jaschke; 1998).)

Even if both A and M are $\sigma(L')$-closed, $A - M$ need not be $\sigma(L')$-closed. So, in general we only have weak duality. The following is a simple application of theorem 3 to the set $A - M$.

Corollary 9 (FTAP) *Let $A - M$ be a cone that contains $\mathbf{1}$, $K_M = (A - M)^*$ its right polar cone, and $\rho = \rho_{A-M}$ the corresponding risk measure. Then*

(i) $A - M \subseteq {}^*K_M$.

(ii) There is no good deal if the set of normalized consistent price systems D_M is nonempty.

(iii)

$$\rho(x) \geq \sup_{\pi \in K_M} \pi(-x)/\pi(\mathbf{1}) \tag{16}$$

with the convention $0/0 = -\infty$.

If $A - M$ is $\sigma(L')$-closed, then (ii) becomes an equivalence and the equality holds in (i) and (iii).

Since $A - M = {}^*K_M$ if and only if $A - M$ is $\sigma(L')$-closed, the business of proving a strong version of the "fundamental theorem of asset pricing" consists in establishing conditions under which $A - M$ is $\sigma(L')$-closed or "changing" M somewhat in order to "complete" $A - M$. Once this is achieved and if also $\mathbf{1}$ is in the radial interior of $A - M$, one gets

$$\begin{aligned} \overline{\pi}_M(z) &= \sup_{\pi \in D_M} \pi(z) \text{ and} \\ \underline{\pi}_M(z) &= \inf_{\pi \in D_M} \pi(z) \end{aligned}$$

which could be called an "extended present value principle".

The practice of pricing OTC contracts and financial derivatives in the past consisted mainly in "calibrating" a complete market model, which yields a set D_M that contains only one price functional $\hat{\pi}$, which is then used for pricing. This is essentially the standard present value principle. As more and more results for incomplete markets are established, we see the future business of pricing OTC contracts as *estimating* a price functional $\hat{\pi}$ by applying *filtering* techniques and taking full account of the *statistical* nature of that process, **and** computing valuation bounds $(\underline{\pi}_M, \overline{\pi}_M)$. Since transaction costs, (conic) trading constraints and "structural market incompleteness" enter the computation of the valuation bounds, the difference $\overline{\pi}_M(x) - \hat{\pi}(x)$ indicates how difficult it is to hedge a single short position in x due to market incompleteness. More precisely, $\overline{\pi}_M(x)$ is the lowest price at which we can install a trading strategy that hedges a short position in x such that the residual risk becomes acceptable (in terms of A). Since a trading desk offering OTC contracts usually has a whole book of contracts partly offsetting their risks, $\overline{\pi}_M(x)$ will usually be an upper bound for the asked price of x in the OTC market place.

5 Examples and Counter Examples of Coherent Risk Measures on the Space of Simple Random Variables

(i) The set of non-negative random variables is obviously a coherent acceptance set. If a set M of cash streams available in the market is given, the corresponding good deal bounds $\overline{\pi}_M$, $\underline{\pi}_M$ are called *arbitrage bounds.*

(ii) Given a price system $\hat{\pi}$, the half-space

$$A = \{x \,|\, \hat{\pi}(x) \geq 0\}$$

is obviously also a coherent acceptance set, although not a strongly relevant one. In fact, $\overline{\pi} = \underline{\pi} = \hat{\pi} = -\rho$, and ρ is linear.

(iii) Given a probability measure P, a threshold t and a power $p \in (1, \infty)$,

$$\mathrm{LPM}_{t,p}(X) = \mathrm{E}_P[((X/(1+r) - t)^-)^p]$$

is a *lower partial moment* of the random variable X. It can be viewed as a "risk measure", but is not coherent. In fact, any coherent risk measure ρ of the form $\rho(X) = \mathrm{E}[g(X)]$ for some real-valued function g is linear, since the translation property implies $g(\alpha) = g(0) - \alpha$.

(iv) Certain pre-orders on sets of probability distributions are known as *stochastic dominance.* They also imply pre-orders on a space of random variables, given a probability measure P. If F^X denotes the cumulative distribution function of the random variable X under P, then first order stochastic dominance is defined as

$$X \geq_1 Y := F^X(t) \leq F^Y(t) \quad \forall t \in \mathbb{R}.$$

Second order stochastic dominance is defined as

$$X \geq_2 Y := \int_{-\infty}^{t} F^X(s)ds \leq \int_{-\infty}^{t} F^Y(s)ds \quad \forall t \in \mathbb{R}.$$

These orderings are not vector orderings on the space of simple random variables and hence do not lead to coherent risk measures.

The following risk measures are discussed in (Artzner et al.; 1998):

(v) *SPAN* (Chicago Mercantile Exchange; 2000) is a margin system developed by the Chicago Mercantile Exchange. It is an example of a finitely generated set K of admissible price systems. The associated risk measure is coherent.

(vi) The *SEC rules* present an example of a finitely generated acceptance set. According to (Artzner et al.; 1998, p.12), the acceptance set is defined as $A = \text{cone}(\{a_1, \dots, a_n\})$ from some "standard acceptable risks" a_j, but A does not contain L^+_{so}. This could easily be made coherent by defining

$$A = \text{cone}(\{a_1, \dots, a_n\}) + L^+_{so}$$

instead.

(vii) Given a probability measure P and a "confidence level" α, the *value at risk* is defined as the negative of the α-quantile of the discounted random variable:

$$\text{VaR}_\alpha(X) = -\inf\{x \mid P\{X/(1+r) \leq x\} > \alpha\},$$

the *tail conditional expectation* is defined as

$$\text{TCE}_\alpha(X) = -\text{E}_P[X/(1+r)] \mid X/(1+r) \leq -\text{VaR}_\alpha(X)], \text{ and}$$

the *worst conditional expectation* is

$$\text{WCE}_\alpha(X) := -\inf_{B:P(B)>\alpha} E_P[X/(1+r) \mid B].$$

Although VaR and TCE satisfy the translation property, positive homogeneity, and monotonicity, they are not coherent as they are not subadditive. WCE is a special case of (9) and hence coherent. Although TCE itself is not coherent, it can be used as an approximation of WCE (Artzner et al.; 1998, Prop. 5.3). *Example.* Consider two independent random variables X and Y taking the value -1 with probability p and 0 otherwise. (For simplicity assume zero interest $r = 0$.) If $p = \alpha \in (0,1)$, then $\text{VaR}_\alpha(X) = \text{VaR}_\alpha(Y) = 0$, but $\text{VaR}_\alpha(X+Y) = 1$. $\text{TCE}_\alpha(X) = \text{TCE}_\alpha(Y) = \alpha$, but $\text{TCE}_\alpha(X+Y) = 2/(2-\alpha) > 2\alpha$.

VaR and LPM are examples of risk measures that are widely used (VaR in banks and LPM in a series of recent papers by Cvitanic, Pham, Leukert and others), but not coherent. They are all based on a single probability measure P. Since P has to be estimated from data, there is a very natural way of getting a whole family of probability measures. Instead of using the best fit in a least squares ("calibration") framework, use the family of probability measures that achieves a certain goodness of fit. Instead of using the probability measure that achieves the maximum likelihood, use the family of probability measures that is above a certain level of likelihood. So most estimation schemes can be used to define a family of "likely" probability measures instead of a point estimate. This is of course related to confidence intervals. Given the wide range of estimation techniques, this produces a wide range of coherent risk measures by (9). Note that these risk measures do not measure *financial risk* but *estimation* or *model risk*.

6 Portfolio Optimization

In this section we will assume that a strongly relevant, coherent acceptance set A and a set of available cash streams M is given, $A - M$ is $\sigma(L')$-closed, and $\mathbf{1} \notin M - A$. Then the strong duality theorem applies, there is no good deal and the set of normalized consistent price systems D_M is not empty. We will also assume that a normalized price system $\hat{\pi} \in D_M$ is estimated. As this can be viewed as the expected present value under an estimated (objective) probability measure, we will call it μ. Let $x_0 \in L$ denote the cash stream generated by the initial position of an investor. (In L_{sm} this can be an initial plan of dynamic trading.) We will denote by ρ the "raw" risk measure corresponding to A.

Now

$$\max_{x-x_0\in M} \quad \{\mu(x) \,|\, \rho(x) \le c_1\} \tag{17}$$

$$\max_{x-x_0\in M} \quad \mu(x) - \lambda\rho(x) \tag{18}$$

$$\min_{x-x_0\in M} \quad \{\rho(x) \,|\, \mu(x) \ge c_2\} \tag{19}$$

are three parameterized optimization problems, which are equivalent under mild conditions on the existence of the Lagrange multiplier λ. Each of these optimization problems describes the (μ, ρ)-*efficient frontier* in M, depending on x_0. If x_0 is a (forced) short position in a contract, then the portfolio optimization problem is the problem of *hedging* the obligation from that contract. Since μ is linear and ρ is convex,

$$U_\lambda(x) = \mu(x) - \lambda\rho(x)$$

can be interpreted as a family of utility functions.

There are a few important economic aspects to note. While both a utility function and a coherent risk measure assign a number to every cash stream, the economic meaning of a utility function is to define a *total ordering* whereas a coherent risk measure is meant to define a *partial ordering* (or pre-order). Second, if $x \in A$ has the economic meaning that "the risk of x is really low", in other words "A is not close to defining a total ordering", then the hedging strategy that achieves the optimum in

$$\overline{\pi}_M(-x_0) = \min_{x\in M} \rho(x + x_0) \tag{20}$$

is usually not done in practice. This can be seen from the fact that (20) is equivalent to (19) with $c_2 = -\infty$.

A prominent class of utility functions in L_{so} is that of *expected utilities*

$$U(x) = \hat{\mathrm{E}}[u(x)]$$

for some concave $u : \mathbb{R} \to \mathbb{R}$. For strictly concave utilities, like $u(x) = \log(x)$, $U(x)$ is not positively homogeneous. In other words, the (μ, ρ) portfolio optimization problem is in this aspect different from the "usual" utility maximization problem. The utility maximization problem is "fully personal", while the

(μ, ρ) portfolio optimization problem is somewhat generic in the sense that the "usual" coherent risk measure is somewhat generic.

Conclusion and Open Questions

We believe that the coherent risk measures introduced by Artzner et al. (1998) are a well-thought-out concept, whereas many of the more widely used risk measures, like VaR and LPM, are not coherent. Moreover, coherent risk measures can be generalized in the form presented here and link well with the established economic theories of arbitrage on the one hand and utility maximization on the other hand. In fact, it seems as if there is room for a "new" economic theory covering coherent risk measures, the associated valuation bounds, and (μ, ρ)-portfolio optimization. Its applications appear to be promising because it is a compromise between the (too) weak assumptions of the arbitrage theory and the (too) individualistic assumptions of utility maximization.

Among the many open questions are:

- What are nontrivial coherent risk measures for stochastic cash streams that are not obvious generalizations of risk measures for random variables?
- Hodges and Neuberger (1989) derived valuation bounds from the principle of (single-agent) utility maximization. By varying the utility function and considering the sup and the inf, resp., of the valuation bounds one gets bounds that are somewhat independent of personal preferences. Under what conditions are these bounds coherent?
- What parts of Markovitz' theory carry over, what is different in (μ, ρ)-optimization?

The Math behind Coherent Risk Measures

This section is intended as a short presentation of the mathematical concepts and results needed to deal with coherent risk measures in infinite-dimensional spaces. The results are extracted from (Dunford and Schwartz; 1958; Köthe; 1960; Kelley and Namioka; 1963; Schaefer; 1966; Jameson; 1970; Kantorowitsch and Akilow; 1978; Yosida; 1980; Bourbaki; 1987; Kamthan and Gupta; 1985; Wong; 1992; Grothendieck; 1992). Where the names of mathematical objects differed, we tended to follow the newer accounts.

6.1 Real Linear Spaces

Recall that a set $A \subseteq L$ is *convex* if for any $x, y \in A$ the *line segment* $[x : y] := \{\lambda x(1-\lambda)y \mid \lambda \in [0,1]\}$ is enclosed in A. Clearly, any intersection of convex sets is convex. The *convex hull* $\langle A \rangle$ of a set A is the intersection of all convex sets containing A. If A and B are convex then the set $A+B := \{a+b \,|\, a \in A, b \in B\}$ is also convex. A set A is called *radial at a point* x if for each $y \neq 0$ exists a $\delta > 0$ s.t. $[x, x + \delta y] \subseteq A$. The set of points at which A is radial is called the *radial interior* of A.[14] A set is called *radially open* if it is radial at all its points. A set is called *radially closed* if its complement is radially open. A set that is radial at 0 is also called *absorbing*. It is obvious that A is radial at x if and only if $A - x$ is absorbing. We call a set *lc-open* if for every $x \in A$ exists a convex, absorbing set U s.t. $x + U \subseteq A$. Equivalently, A is lc-open if $A - x$ contains a convex, absorbing set for all $x \in A$. A set is called *lc-closed* if its complement is lc-open.

Lemma 10 *(i) Every lc-open set is radially open.*

(ii) A set is radially closed if and only if its intersection with any line is closed (in the topology induced in the line by the real numbers).

(iii) The radial interior of a convex set is convex and is its own radial interior.

(iv) A convex, radially open set is lc-open.

(v) A is lc-closed if and only if $A = \cap_U (A + U)$, where the intersection is taken over all convex, absorbing sets U.

Proof. (i) Trivial.
(ii) Easy.
(iii) (Kelley and Namioka; 1963, p.15)
(iv) If A is convex and radially open, then $A - x$ is a convex, absorbing set for all $x \in A$.
(v) Since $A \subseteq \cap_U (A + U)$, the r.h.s. condition does *not* hold if and only if

$$\begin{aligned} & \exists x \notin A \; \forall U \text{ convex, absorbing s.t. } x \in A + U \\ \iff \; & \exists x \notin A \; \forall U \text{ convex, absorbing s.t. } -U \cap (A - x) \neq \emptyset. \end{aligned}$$

Since U is convex and absorbing if and only if $-U$ is convex and absorbing, this is equivalent to the fact that the complement of A is not lc-open. □

A set A is called *balanced (circled, equilibré)* if $\alpha A \subseteq A$ for all $|\alpha| \leq 1$. The *Minkowski functional*[15] of a set U

$$p_U(x) := \inf\{\alpha \,|\, \alpha^{-1} x \in U, \alpha > 0\}$$

is finite for every absorbing set U. p_U is *positively homogeneous* ($p_U(\alpha x) = \alpha p(x) \;\; \forall \alpha > 0$). If U is convex then p_U is *sub-additive* ($p_U(x + y) \leq p_U(x) + p_U(y)$). If U is balanced then p_U is *absolutely homogeneous* ($p(\alpha x) = |\alpha| p(x) \;\; \forall \alpha \in \mathbb{R}$). Hence p_U is a *semi-norm*, i.e. sub-additive and absolutely

[14] This is also called the *linear interior* or the *algebraic interior*.
[15] This is also called *support function*, *distance function*, or *gauge*.

homogeneous, if U is convex, absorbing, and balanced. On the other hand, every semi-norm p defines a convex, absorbing, balanced set by

$$U_p := \{x \in L \,|\, p(x) \leq 1\}.$$

If U is convex, absorbing, then $\{x \in L \,|\, p(x) < 1\}$ is its radial interior and $\{x \in L \,|\, p(x) \leq 1\}$ is its radial closure.

A is a *cone* if $A + A \subseteq A$ and $\alpha A \subseteq A$ for all $\alpha \geq 0$. Clearly, the intersection of cones is a cone and every cone is convex. A cone A defines a vector ordering $\succeq$ by $x \succeq y \overset{\text{df}}{=} x - y \in A$. Conversely, $\{x \,|\, x \succeq 0\}$ is a cone for any vector ordering $\succeq$. Each linear subspace is a cone. $A - A$ and $A \cap (-A)$ are subspaces. A is said to *generate* L, if $A - A = L$. The order defined by A is *antisymmetric* ($x \succeq y, y \succeq x \implies x = y$) if and only if $A \cap (-A) = \{0\}$. A linear functional $\pi \in L^\times$ is said to *separate* two sets A and B if $\sup \pi(A) \leq \inf \pi(B)$. π is *strictly separating* A and B if $\sup \pi(A) < \inf \pi(B)$.

For any set $A \subseteq L$,

$$A^* = \{\pi \in L^\times \,|\, \pi(x) \geq -1 \quad \forall x \in A\}$$

is called the *right polar set* of A. For any set $K \subseteq L^\times$, the set

$$^*K = \{x \in L \,|\, \pi(x) \geq -1 \quad \forall \pi \in K\}$$

is called the *left polar set* of K. It is easily verified that A^* is convex and if A is a cone then

$$A^* = \{\pi \in L^\times \,|\, \pi(x) \geq 0 \quad \forall x \in A\}$$

holds and A^* is a cone. The same holds for the left polar set and cone. Obviously, $A \subseteq {}^*(A^*)$ holds for any set A. For the inverse conclusion, one has to show that $x \notin A$ implies $x \notin {}^*(A^*)$. By definition of the polar set, this is the same as that for each $x \notin A$ exists a $\pi \in A^*$ such that $\pi(x) < -1$. If $x \neq 0$, this is the same as that x can be strictly separated from A.

Corollary 11 $A = {}^*(A^*)$ *holds if and only if every point from the complement of* A *can be strictly separated from* A *by a linear functional and* $A \ni 0$.

We cite here two separating hyperplane theorems from (Kelley and Namioka; 1963, pp.22).

Theorem 12 (Weak Separation Theorem) *If* F *and* G *are convex sets and* F *is radial at some point, then* F *and* G *can be separated if and only if the intersection of* G *and the radial interior of* F *is empty.*

Theorem 13 (Strong Separation Theorem) *Two convex sets* F *and* G *can be strictly separated if and only if there exists a convex, absorbing set* U *such that* $(F + U) \cap G = \emptyset$.

Applying the strong separation theorem to $F := \{x\}$ for $x \notin A$ and $G := A$ in corollary 11, we get

Corollary 14 $A = {}^*(A^*)$ *holds if and only if* A *is convex, lc-closed, and* $A \ni 0$.

6.2 Linear Topological Spaces

Recall that a family $\mathcal{T}$ of sets in L is a *topology* if $\mathcal{T}$ contains (i) $\emptyset$ and L, (ii) any union of sets in $\mathcal{T}$, and (iii) any finite intersection of sets in $\mathcal{T}$. The sets in $\mathcal{T}$ are called $\mathcal{T}$*-open* and $(L, \mathcal{T})$ a *topological space.* A set U is a $\mathcal{T}$*-neigborhood* of a point x if there is such a $\mathcal{T}$-open set G, that $x \in G \subseteq U$. A point x is in the $\mathcal{T}$*-interior* of a set A if A is a neighborhood of x. A set is $\mathcal{T}$*-closed* if its complement is $\mathcal{T}$-open. A mapping f from a topological space $(L_1, \mathcal{T}_1)$ to another topological space $(L_2, \mathcal{T}_2)$ is *continuous* if $f^{-1}(G)$ is $\mathcal{T}_1$-open for any $\mathcal{T}_2$-open set G. On a cartesian product $L_1 \times L_2$ of two topological spaces a *product topology* is defined by calling sets $(\mathcal{T}_1 \times \mathcal{T}_2)$-open if they can be represented as the union of sets of the form $G_1 \times G_2$, where $G_1 \in \mathcal{T}_1$ and $G_2 \in \mathcal{T}_2$. A topology on a linear space is called a *vector topology* if the vector translation $(x, y) \mapsto x + y$ and the scalar multiplication $(\alpha, x) \mapsto \alpha x$ are continuous (in their respective product topologies). A linear space with a vector topology is called a *linear topological space.*

Because of the continuity of the vector translation, U is a neighborhood of x if and only if $U - x$ is a neighborhood of 0. So it is convenient to describe a vector topology in terms of its neighborhoods of 0. A family $\mathcal{U}$ of neighborhoods of 0 is called a *local base* if each neighborhood of 0 contains an $U \in \mathcal{U}$.

Theorem 15 *If $\mathcal{U}$ is a local base of a vector topology, then*

(i) for all $U \in \mathcal{U}$ exists a $V \in \mathcal{U}$: $V + V \subseteq U$,

(ii) for all $U \in \mathcal{U}$ exists $V \in \mathcal{U}$ s.t. $\forall \alpha, |\alpha| \leq 1$: $\alpha V \subseteq U$,

(iii) all elements in $\mathcal{U}$ are absorbing sets, and

(iv) for any two sets $U, V \in \mathcal{U}$ exists a set $W \in \mathcal{U}$: $W \subseteq U \cap V$.

A local base can be chosen such that instead of (ii) the sets $U \in \mathcal{U}$ themselves are balanced. If a family of sets containing 0 satisfies (i)-(iv), then it induces a vector topology by defining a set G open if for all $x \in G$ exists a $U \in \mathcal{U}$ s.t. $x + U \subseteq G$. (The vector topology is Hausdorff *if and only if $\cap_{U \in \mathcal{U}} U = 0$.)*

Proof. (Köthe; 1960, p.149), (Kelley and Namioka; 1963),(Kantorowitsch and Akilow; 1978, p.317), (Schaefer; 1966, p.14).

A linear topological space $(L, \mathcal{T})$ is called *locally convex,* if every $\mathcal{T}$-neighborhood of 0 contains a convex $\mathcal{T}$-neighborhood of 0. A closed, convex, absorbing, balanced set is called a *barrel.* A locally convex space $(L, \mathcal{T})$ is a *barrel space* if each of its barrels is a neighborhood of 0.

Proposition 16 *In a linear space L, the family of convex, absorbing sets is a local base for a vector topology. L is a barrel space with this topology.*

Proof.

(i) U convex $\implies \frac{1}{2}U + \frac{1}{2}U \subseteq U$.

(ii) $V := U \cap (-U)$ is balanced, convex, and absorbing for every convex, absorbing set U.

(iii) is automatically fulfilled.

(iv) $W := U \cap V$ is convex, absorbing for all convex, absorbing sets U and V.

□

The lc-topology is the finest locally convex topology. Since $U \cap (-U)$ is balanced, convex, and absorbing for every convex, absorbing set U, the family of all convex, absorbing, balanced sets is also a local base for the lc-topology. Similarly it can be shown that if $\mathcal{U}$ is a family of convex, absorbing, balanced sets then the family of sets

$$\{\cap_{k=1}^{n} \lambda_k U_k \,|\, \lambda_k > 0, U_k \in \mathcal{U}, n \in \mathbb{N}\}$$

is a local base for a locally convex vector topology (Kantorowitsch and Akilow; 1978, p.327). Because of the one-to-one correspondence between radially closed, convex, absorbing, balanced sets and semi-norms, any family of semi-norms generates a locally convex topology. (The topology is Hausdorff if and only if the family of semi-norms satisfies the *axiom of separation*: for any $x \neq 0$ exists a semi-norm p from the family s.t. $p(x) > 0$. Such families of semi-norms are also called *multi-norms.*) On the other hand, any locally convex topology is generated by the family of the Minkowski functionals of the convex, absorbing, balanced sets that are 0-neighborhoods.

Proposition 17 *(i) If $(L, \mathcal{T})$ is a linear topological space, then any $\mathcal{T}$-open set is radially open.*

(ii) If $(L, \mathcal{T})$ is a locally convex space, then any $\mathcal{T}$-open set is lc-open.

Proof.

(i) A set A is radially open if and only if $\{\delta \,|\, x + \delta y \in A\}$ is open for all $x, y \in L, y \neq 0$. But this set equals $f^{-1}(A)$ for $f(\delta) = x + \delta y$, which is $\mathcal{T}$-continuous since $\mathcal{T}$ is a vector topology.

(ii) Every $\mathcal{T}$-neighborhood of 0 contains a convex $\mathcal{T}$-neighborhood V. V is absorbing because of (i). Hence, every $\mathcal{T}$-neighborhood contains a convex, absorbing set.

In a linear topological space, the closure of a linear subspace is a linear subspace (Kantorowitsch and Akilow; 1978, p.319). From this follows that the null space $\{x \,|\, \pi(x) = 0\}$ of a non-zero linear functional π is either closed or dense in L. This implies that if π separates two sets one of which has a non-empty interior, then π is continuous. Moreover, a discontinuous linear functional takes any value on any neigborhood of 0.

Proposition 18 *If A is a convex set in a topological vector space $(L, \mathcal{T})$ and A has a non-empty $\mathcal{T}$-interior, then the radial and $\mathcal{T}$-interior coincide. The same holds for the radial and $\mathcal{T}$-closure.*

Proof. (Grothendieck; 1992, p.52), (Dunford and Schwartz; 1958, p.413)

Proposition 19 *If A is a convex, radially closed set in a vector space and it is radial at one of its points, then A is lc-closed.*

Proof. Since A is convex and radial at one point it has a non-empty lc-interior. Then the previous proposition applies. □

6.3 The Bipolar Theorem

A pair of vector spaces (L, L') and a bilinear form $\langle ., . \rangle$ represent a *dual system* if $\langle x', x \rangle = 0$ for all $x \in L$ implies $x' = 0$ and $\langle x', x \rangle = 0$ for all $x' \in L'$ implies $x = 0$. If L' is a *total subspace* of $L^\times$, i.e. $\pi(x) = 0$ for all $\pi \in L'$ implies $x = 0$, then (L, L') is a dual system with the bilinear form $\langle \pi, x \rangle = \pi(x)$. The $\sigma(L')$-topology on L is the coarsest topology for which the linear functionals $x \mapsto \pi(x)$ are continuous for all $\pi \in L'$. Similarly, the $\sigma(L)$-topology on L' is the coarsest topology for which the linear functionals $\pi \mapsto \pi(x)$ are continuous for all $x \in L$. As a matter of fact, the $\sigma(L')$-topology is generated by the family of semi-norms $p_\pi(x) = |\pi(x)|, \pi \in L'$, so this topology is a locally convex topology.

(Re-) Define the right polar set

$$A^* = \{\pi \in L' \,|\, \pi(x) \geq -1 \quad \forall x \in A\}.$$

of a set A.

Theorem 20 (Bipolar Theorem) $A = {}^*(A^*)$ *holds for a set A in the dual system (L, L') if and only if A is $\sigma(L')$-closed, convex, and $A \ni 0$.*

Proof. Let $A = {}^*(A^*)$. Then for all $x \notin A, x \neq 0$ exists a $\pi_x \in L'$ that strictly separates A and $\{x\}$ s.t. $\inf \pi_x(A) \geq -1$ and $\pi_x < -1$. In other words,

$$A = \bigcap_{x \notin A, x \neq 0} \{y \,|\, \pi_x(y) \geq -1\}.$$

Since A is the intersection of $\sigma(L')$-closed half-spaces, A is $\sigma(L')$-closed. Any polar set is convex and contains 0, so the first part is done.

For the other direction we have to show that for all $x \notin A$ exists a $\pi \in L'$ that strictly separates A and $\{x\}$. Since A is $\sigma(L')$-closed, there exists a $\sigma(L')$-neighborhood U_x for every $x \notin A$ with $x \in U_x \subseteq A^c$. Since $\sigma(L')$ is a locally convex topology, U_x can be chosen convex. The weak separating hyperplane theorem implies that there is a $\pi \in L^\times$ weakly separating A and U_x. For U_x has a non-empty $\sigma(L')$-interior, π is in fact $\sigma(L')$-continuous, i.e. $\pi \in L'$. π strictly separates A and $\{x\}$. □

Corollary 21 *If A is convex, it is lc-closed if and only if it is $\sigma(L^\times)$-closed.*

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